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A systematical way to find breather lattice solutions to the positive mKdV equation

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Abstract

In this paper, dependent and independent variable transformations are introduced to solve the positive mKdV equation systematically by using knowledge of elliptic equation and Jacobian elliptic functions. It is shown that different kinds of solutions can be obtained to the positive mKdV equation, including many kinds of breather lattice solutions.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Among the soliton bearing nonlinear equations, the modified Korteweg-de Vries (mKdV) equation is of special interest [1, 2]; for it possesses rich solutions, such as solitary solutions [1–4], periodic solutions [3–6], breather solutions [1, 2, 7, 8], and breather lattice solutions [7, 8]. A particularly interesting type of solution is the so-called breather kind of solution, usually this kind of solution is unavailable and such solutions have to be solved numerically [7]. In some cases, however, the analytical expressions in closed form can be found, such as the breather lattice solution for the sine-Gordon equation [9] and for the mKdV equation [7, 8].

In [7–9], Kevrekidis's research group has applied some ansatzs to obtain the breather lattice solutions to the mKdV equation and the sine-Gordon equation. The aim of the present paper is to present the breather lattice solutions of the positive mKdV equation in a systematical way. Based on the introduced transformations, we will show systematical results about these breather-type solutions for the positive mKdV equation by using the knowledge of elliptic equation and Jacobian elliptic functions [3, 4, 10–12].

2. Breather lattice solutions to the positive mKdV equation

The positive mKdV equation reads [7, 8]

$$u_t + 6u^2 u_x + u_{xxx} = 0. \tag{1}$$

In order to solve the sine-Gordon-type equation, certain dependent or independent variable transformations must be introduced. For example, the dependent variable transformation

$$v = 2\tan^{-1}\phi \quad \text{or} \quad \phi = \tan\frac{v}{2} \tag{2}$$

has been introduced in [1, 2, 13] to solve the sine-Gordon equation and the double sine-Gordon equation.

So, in order to derive the breather lattice solutions to the positive mKdV equation (1), first of all we introduce another dependent variable transformation

$$u = v_x \tag{3}$$

to build the bridge between (1) and (2), and then ϕ satisfies the equation [7, 8]

$$(1+\phi^2)(\phi_t + \phi_{xxx}) + 6\phi_x\left(\phi_x^2 - \phi\phi_{xx}\right) = 0, \tag{4}$$

which can be taken as another form of the positive mKdV equation (1).

Next, we introduce independent variable transformation

$$\xi = ax + bt + \xi_0, \qquad \eta = cx + \mathrm{d}t + \eta_0, \tag{5}$$

where ξ_0 and η_0 are two constants.

Considering the transformation (5), equation (4) can be rewritten as

$$(1+\phi^2)[(b\phi_\xi + d\phi_\eta) + (a^3\phi_{\xi\xi\xi} + 3a^2c\phi_{\xi\xi\eta} + 3ac^2\phi_{\xi\eta\eta} + c^3\phi_{\eta\eta\eta})] \\ + 6(a\phi_\xi + c\phi_\eta)[(a\phi_\xi + c\phi_\eta)^2 - \phi(a^2\phi_{\xi\xi} + 2ac\phi_{\xi\eta} + c^2\phi_{\eta\eta})] = 0. \tag{6}$$

Compared to the transformation given in [1, 2], transformation (5) has less constraints; of course, this will let us have more different types of solutions to the positive mKdV equation (1).

Inspired by the transformation given in [2] and the results in [7, 8], we choose dependent variable transformation

$$\phi = \alpha U(\xi)V(\eta), \tag{7}$$

where α is a constant amplitude to be determined, U and V satisfy the following elliptic equations:

$$U_\xi^2 = -n^2U^4 + p_1U^2 + q_1, \qquad V_\eta^2 = -\beta n^2V^4 + p_2V^2 + q_2, \tag{8}$$

where β, n^2, p_1, p_2, q_1 and q_2 are determined constants for the determined analytical expressions of U and V. Here, one point must be stressed is that the introduction of β will let us have more choices to obtain different kinds of solutions to the positive mKdV equation.

Remark 1. Through the successive dependent variable transformations (2), (3) and (7), double independent variable transformation (5), the positive mKdV equation (1) is mapped to the coupled elliptic equations (8). So, the solutions to the positive mKdV equation (1) are the so-called breather lattice solutions, which can be periodic in two directions (details can be found in the next parts). Different from what we have done in this paper, in [5], the mKdV equation is solved directly or through a fractional dependent transformation, where only periodic solutions, which are only periodic in a specific direction, expressed in terms of Jacobi elliptic functions were obtained; similarly, in [6], the mKdV equation is solved by using the projective Riccati equations as intermediate transformation, where only some solitary wave solutions were derived.

Remark 2. Through the dependent transformation (7), the independent variable transformation (5) and the coupled elliptic equations (8), we can obtained 18 families of breather lattice solutions by a single way without using the ansatz proposed in [7, 8], where only 3 families of breather lattice solutions were derived. Details can be found in the next parts.

Substituting (7) and (8) into (6) yields the following algebraic equations:

$$b + p_1 a^3 + 3p_2 a c^2 = 0, \tag{9a}$$

$$\beta n^2 c^2 - q_1 \alpha^2 a^2 = 0, \tag{9b}$$

$$q_2 \alpha^2 c^2 - n^2 a^2 = 0, \tag{9c}$$

$$d + 3p_1 a^2 c + p_2 c^3 = 0, \tag{9d}$$

from which we can determine

$$\begin{aligned} &\alpha^4 = \frac{\beta n^4}{q_1 q_2}, \qquad \frac{a^2}{c^2} = \frac{\beta n^2}{q_1 \alpha^2} = \frac{q_2 \alpha^2}{n^2}, \\ &b = -a(p_1 a^2 + 3p_2 c^2), \qquad d = -c(3p_1 a^2 + p_2 c^2). \end{aligned} \tag{10}$$

From (10), it is obvious that the determined constants in (8) must satisfy the following constraints:

$$\frac{\beta}{q_1 q_2} > 0, \qquad \frac{q_2}{n^2} > 0, \qquad \frac{\beta n^2}{q_1} > 0, \tag{11}$$

this implies that not all combinations of Jacobi elliptic functions are solutions to the positive mKdV equation (1) under the above-mentioned transformations, only the combination of a couple of the Jacobi elliptic functions satisfies the constraint (11), it can be a solution to the positive mKdV equation (1). Actually, there exist only 18 families of these kinds of combinations, we will address them in details.

Case 1. When $U = \operatorname{sn}(\xi, k)$ and $V = \operatorname{dn}(\eta, m)$, where $\operatorname{sn}(\xi, k)$ and $\operatorname{dn}(\eta, m)$ are the Jacobi sine elliptic function and the Jacobi elliptic function of the third kind, respectively, and k and m are their modulus [10–12]. Then from (8), we have

$$\begin{aligned} &n^2 = -k^2, \qquad p_1 = -(1+k^2), \qquad q_1 = 1, \\ &\beta n^2 = 1, \qquad p_2 = 2 - m^2, \qquad q_2 = -(1-m^2). \end{aligned} \tag{12}$$

Substituting (12) into (10), the parameters can be determined as

$$\begin{aligned} &\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}}, \qquad \frac{b}{a} = [a^2(1+k^2) - 3c^2(2-m^2)], \\ &\frac{d}{c} = [3a^2(1+k^2) - c^2(2-m^2)], \qquad \alpha = \pm\left[\frac{k^2}{1-m^2}\right]^{\frac{1}{4}}, \end{aligned} \tag{13}$$

then the solution to the positive mKdV equation (4) is

$$\phi_1 = \pm\left[\frac{k^2}{1-m^2}\right]^{\frac{1}{4}} [\operatorname{sn}(\xi, k)\operatorname{dn}(\eta, m)], \tag{14}$$

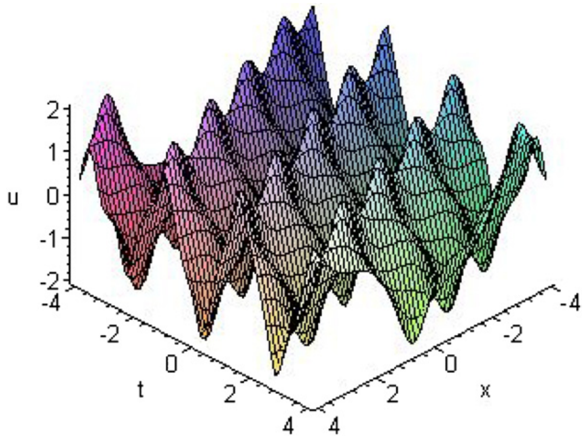


Figure 1. The graphical presentation shows the spacetime evolution of the breather lattice solution of equations (13) and (14), where the parameters are chosen as $a = 1, c = 1, m = 0.8, \xi_0 = \eta_0 = 0$, from which other parameters can be determined as $b = -2.72, d = 2.72, k = 0.6$ and $\alpha = 1$.

which is a kind of breather lattice solution given in [7, 8], and when $m \to 0$, $\mathrm{dn}(\eta, m) \to 1$, it turns to be a periodic wave solution

$$\phi_{1'} = \pm\sqrt{k}\,\mathrm{sn}(\xi, k). \tag{15}$$

Figure 1 shows the evolution of the breather lattice solution with the periodic characteristics in both spatial and temporal directions, while for the normal breather solution that has the periodic characteristics just in a specific direction.

Case 2. When $U = \mathrm{cn}(\xi, k)$ and $V = \mathrm{cn}(\eta, m)$, where $\mathrm{cn}(\xi, k)$ and $\mathrm{cn}(\eta, m)$ are the Jacobi cosine elliptic function [10–12]. And then from (8), we have

$$\begin{aligned} &n^2 = k^2, && p_1 = 2k^2 - 1, && q_1 = 1 - k^2, \\ &\beta n^2 = m^2, && p_2 = 2m^2 - 1, && q_2 = 1 - m^2. \end{aligned} \tag{16}$$

Substituting (16) into (10), one has

$$\begin{aligned} &\frac{a^2}{c^2} = \frac{m}{k}\sqrt{\frac{1-m^2}{1-k^2}}, && \frac{b}{a} = [a^2(1-2k^2) + 3c^2(1-2m^2)], \\ &\frac{d}{c} = [3a^2(1-2k^2) + c^2(1-2m^2)], && \alpha = \pm\left[\frac{k^2m^2}{(1-k^2)(1-m^2)}\right]^{\frac{1}{4}}, \end{aligned} \tag{17}$$

then the solution to the positive mKdV equation (4) is

$$\phi_2 = \pm\left[\frac{k^2m^2}{(1-k^2)(1-m^2)}\right]^{\frac{1}{4}} [\mathrm{cn}(\xi, k)\mathrm{cn}(\eta, m)], \tag{18}$$

which is another kind of breather lattice solution given in [8].

Case 3. When $U = \mathrm{sc}(\xi, k) = \frac{\mathrm{sn}(\xi,k)}{\mathrm{cn}(\xi,k)}$ and $V = \mathrm{dn}(\eta, m)$. And then from (8), we have

$$\begin{aligned} &n^2 = -(1 - k^2), && p_1 = 2 - k^2, && q_1 = 1, \\ &\beta n^2 = 1, && p_2 = 2 - m^2, && q_2 = -(1 - m^2). \end{aligned} \tag{19}$$

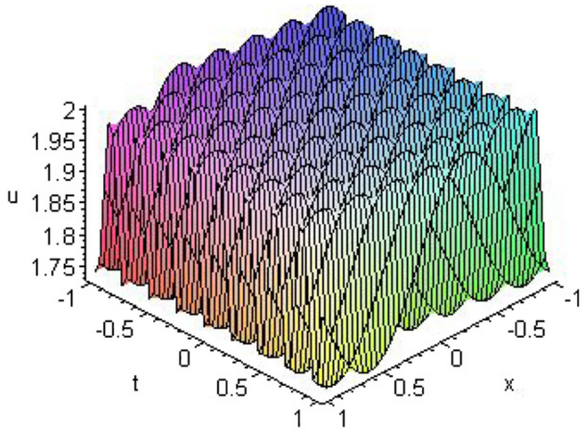


Figure 2. The graphical presentation shows the spacetime evolution of the breather lattice solution of equations (20) and (21), where the parameters are chosen as $a = 1, c = 1, m = 0.5, \xi_0 = \eta_0 = 0$, from which other parameters can be determined as $b = 7, d = 7, k = 0.5$ and $\alpha = 1$.

Substituting (19) into (10), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{1-k^2}}, \qquad \frac{b}{a} = -[a^2(2-k^2) + 3c^2(2-m^2)],$$
$$\frac{d}{c} = -[3a^2(2-k^2) + c^2(2-m^2)], \qquad \alpha = \pm\left[\frac{1-k^2}{1-m^2}\right]^{\frac{1}{4}}, \tag{20}$$

then the solution to the positive mKdV equation (4) is

$$\phi_3 = \pm\left[\frac{1-k^2}{1-m^2}\right]^{\frac{1}{4}} [\mathrm{sc}(\xi, k)\,\mathrm{dn}(\eta, m)], \tag{21}$$

which is the third kind of breather lattice solution given in [8].

It is obvious that figure 2 describes a different kind of breather lattice solution from that given in figure 1. Compared to figure 1, the period in figure 1 is much smaller in both spatial and temporal directions.

Besides the above three kinds of breather lattice solutions, there still exist 15 kinds of breather lattice solutions that have not been reported in the literature, next we will show their details.

Case 4. When $U = \mathrm{cn}(\xi, k)$ and $V = \mathrm{sd}(\eta, m) = \frac{\mathrm{sn}(\eta,m)}{\mathrm{dn}(\eta,m)}$. And then from (8), we have

$$n^2 = k^2, \qquad p_1 = 2k^2 - 1, \qquad q_1 = 1 - k^2,$$
$$\beta n^2 = m^2(1-m^2), \qquad p_2 = 2m^2 - 1, \qquad q_2 = 1. \tag{22}$$

Substituting (22) into (10), the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}\sqrt{\frac{1-m^2}{1-k^2}}, \qquad \frac{b}{a} = [a^2(1-2k^2) + 3c^2(1-2m^2)],$$
$$\frac{d}{c} = [3a^2(1-2k^2) + c^2(1-2m^2)], \qquad \alpha = \pm\left[\frac{k^2m^2(1-m^2)}{1-k^2}\right]^{\frac{1}{4}}, \tag{23}$$

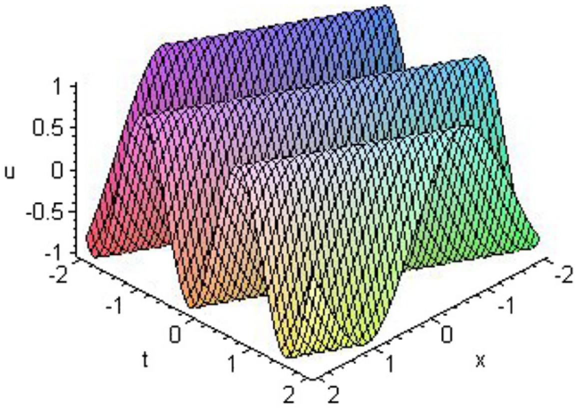


Figure 3. The graphical presentation shows the spacetime evolution of the breather lattice solution of equations (23) and (24), where the parameters are chosen as $a = 1$, $c = 1$, $m = 0.5$, $\xi_0 = \eta_0 = 0$, from which other parameters can be determined as $b = 2$, $d = 2$, $k = 0.5$ and $\alpha = \frac{1}{2}$.

then the solution to the positive mKdV equation (4) is

$$\phi_4 = \pm\left[\frac{k^2m^2(1-m^2)}{1-k^2}\right]^{\frac{1}{4}} [\mathrm{cn}(\xi, k)\mathrm{sd}(\eta, m)]. \tag{24}$$

The periodic characteristics in figure 3 are different from what we see in figures 1 and 2, here the periodic characteristics are obvious only in a specific direction, but much weaker in other directions.

Case 5. When $U = \mathrm{nc}(\xi, k) = \frac{1}{\mathrm{cn}(\xi,k)}$ and $V = \mathrm{ds}(\eta, m) = \frac{\mathrm{dn}(\eta,m)}{\mathrm{sn}(\eta,m)}$. And then from (8), we have

$$\begin{aligned} &n^2 = -(1-k^2), && p_1 = 2k^2 - 1, && q_1 = -k^2, \\ &\beta n^2 = -1, && p_2 = 2m^2 - 1, && q_2 = -m^2(1-m^2). \end{aligned} \tag{25}$$

Substituting (25) into (10), the parameters can be determined as

$$\begin{aligned} &\frac{a^2}{c^2} = \frac{m}{k}\sqrt{\frac{1-m^2}{1-k^2}}, && \frac{b}{a} = [a^2(1-2k^2) + 3c^2(1-2m^2)], \\ &\frac{d}{c} = [3a^2(1-2k^2) + c^2(1-2m^2)], && \alpha = \pm\left[\frac{1-k^2}{k^2m^2(1-m^2)}\right]^{\frac{1}{4}}, \end{aligned} \tag{26}$$

then the solution to the positive mKdV equation (4) is

$$\phi_5 = \pm\left[\frac{1-k^2}{k^2m^2(1-m^2)}\right]^{\frac{1}{4}} [\mathrm{nc}(\xi, k)\,\mathrm{ds}(\eta, m)]. \tag{27}$$

Case 6. When $U = \mathrm{dn}(\xi, k)$ and $V = \mathrm{cs}(\eta, m) = \frac{\mathrm{cn}(\eta,m)}{\mathrm{sn}(\eta,m)}$. And then from (8), we have

$$\begin{aligned} &n^2 = 1, && p_1 = 2 - k^2, && q_1 = -(1-k^2), \\ &\beta n^2 = -1, && p_2 = 2 - m^2, && q_2 = 1 - m^2. \end{aligned} \tag{28}$$

Substituting (28) into (10), the parameters can be determined as

$$\frac{a^2}{c^2}=\sqrt{\frac{1-m^2}{1-k^2}},\qquad \frac{b}{a}=-[a^2(2-k^2)+3c^2(2-m^2)],$$
$$\frac{d}{c}=-[3a^2(2-k^2)+c^2(2-m^2)],\qquad \alpha=\pm\left[\frac{1}{(1-k^2)(1-m^2)}\right]^{\frac{1}{4}}, \tag{29}$$

then the solution to the positive mKdV equation (4) is

$$\phi_6=\pm\left[\frac{1}{(1-k^2)(1-m^2)}\right]^{\frac{1}{4}}[\mathrm{dn}(\xi,k)\mathrm{cs}(\eta,m)]. \tag{30}$$

Case 7. When $U=\mathrm{nd}(\xi,k)=\frac{1}{\mathrm{dn}(\xi,k)}$ and $V=\mathrm{cs}(\eta,m)=\frac{\mathrm{cn}(\eta,m)}{\mathrm{sn}(\eta,m)}$. And then from (8), we have

$$n^2=1-k^2,\qquad p_1=2-k^2,\qquad q_1=-1,$$
$$\beta n^2=-1,\qquad p_2=2-m^2,\qquad q_2=1-m^2. \tag{31}$$

Substituting (31) into (10), the parameters can be determined as

$$\frac{a^2}{c^2}=\sqrt{\frac{1-m^2}{1-k^2}},\qquad \frac{b}{a}=-[a^2(2-k^2)+3c^2(2-m^2)],$$
$$\frac{d}{c}=-[3a^2(2-k^2)+c^2(2-m^2)],\qquad \alpha=\pm\left[\frac{1-k^2}{1-m^2}\right]^{\frac{1}{4}}, \tag{32}$$

then the solution to the positive mKdV equation (4) is

$$\phi_7=\pm\left[\frac{1-k^2}{1-m^2}\right]^{\frac{1}{4}}[\mathrm{nd}(\xi,k)\mathrm{cs}(\eta,m)]. \tag{33}$$

Case 8. When $U=\mathrm{nd}(\xi,k)$ and $V=\mathrm{sc}(\eta,m)$. And then from (8), we have

$$n^2=1-k^2,\qquad p_1=2-k^2,\qquad q_1=-1,$$
$$\beta n^2=-(1-m^2),\qquad p_2=2-m^2,\qquad q_2=1. \tag{34}$$

Substituting (34) into (10), the parameters can be determined as

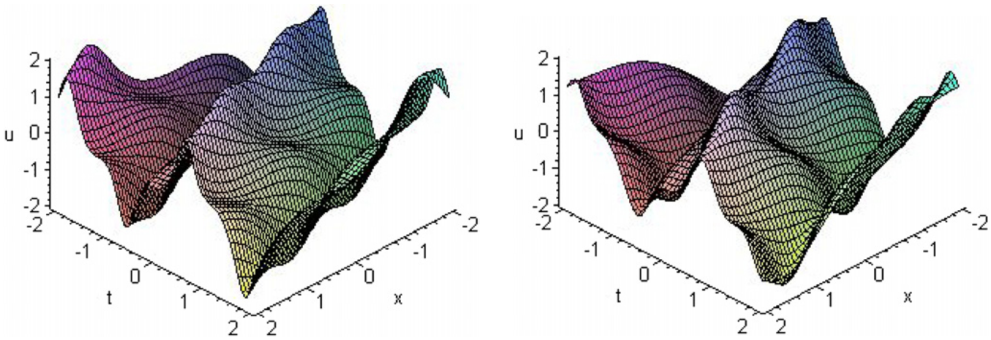
$$\frac{a^2}{c^2}=\sqrt{\frac{1-m^2}{1-k^2}},\qquad \frac{b}{a}=-[a^2(2-k^2)+3c^2(2-m^2)],$$
$$\frac{d}{c}=-[3a^2(2-k^2)+c^2(2-m^2)],\qquad \alpha=\pm[(1-k^2)(1-m^2)]^{\frac{1}{4}}, \tag{35}$$

then the solution to the positive mKdV equation (4) is

$$\phi_8=\pm[(1-k^2)(1-m^2)]^{\frac{1}{4}}[\mathrm{nd}(\xi,k)\mathrm{sc}(\eta,m)]. \tag{36}$$

Case 9. When $U=\mathrm{nd}(\xi,k)$ and $V=\mathrm{sn}(\eta,m)$. And then from (8), we have

$$n^2=1-k^2,\qquad p_1=2-k^2,\qquad q_1=-1,$$
$$\beta n^2=-m^2,\qquad p_2=-(1+m^2),\qquad q_2=1. \tag{37}$$



(*a*) (*b*)

Figure 4. The left panel shows the spacetime evolution of the breather lattice solution of equations (38) and (39), where the parameters are chosen as $a = 1, c = 1, m = 0.8, \xi_0 = \eta_0 = 0$, from which other parameters can be determined as $b = 3.28, d = -3.28, k = 0.6$ and $\alpha = 0.8$. While for the right panel, the parameters are chosen as $a = 1, c = 1, m = 0.6, \xi_0 = \eta_0 = 0$, from which other parameters can be determined as $b = 2.72, d = -2.72, k = 0.8$ and $\alpha = 0.6$.

Substituting (37) into (10), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{m^2}{1-k^2}}, \qquad \frac{b}{a} = -[a^2(2-k^2) - 3c^2(1+m^2)],$$
$$\frac{d}{c} = -[3a^2(2-k^2) - c^2(1+m^2)], \qquad \alpha = \pm[(1-k^2)m^2]^{\frac{1}{4}}, \tag{38}$$

then the solution to the positive mKdV equation (4) is

$$\phi_9 = \pm[(1-k^2)m^2]^{\frac{1}{4}}[\mathrm{nd}(\xi, k)\mathrm{sn}(\eta, m)]. \tag{39}$$

From figure 4, it is obvious that for different values of m and k the same breather lattice solution will also show different characteristics, small or large. Especially, when m and k take their limiting values, the behaviour will be quite different from that given in figure 4.

Case 10. When $U = \mathrm{nd}(\xi, k)$ and $V = \mathrm{cd}(\eta, m) = \frac{\mathrm{cn}(\eta,m)}{\mathrm{dn}(\eta,m)}$. And then from (8), we have

$$n^2 = 1-k^2, \qquad p_1 = 2-k^2, \qquad q_1 = -1,$$
$$\beta n^2 = -m^2, \qquad p_2 = -(1+m^2), \qquad q_2 = 1. \tag{40}$$

Substituting (40) into (10), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{m^2}{1-k^2}}, \qquad \frac{b}{a} = -[a^2(2-k^2) - 3c^2(1+m^2)],$$
$$\frac{d}{c} = -[3a^2(2-k^2) - c^2(1+m^2)], \qquad \alpha = \pm[(1-k^2)m^2]^{\frac{1}{4}}, \tag{41}$$

then the solution to the positive mKdV equation (4) is

$$\phi_{10} = \pm[(1-k^2)m^2]^{\frac{1}{4}}[\mathrm{nd}(\xi, k)\mathrm{cd}(\eta, m)]. \tag{42}$$

Case 11. When $U = \mathrm{nd}(\xi, k)$ and $V = \mathrm{ns}(\eta, m) = \frac{1}{\mathrm{sn}(\eta,m)}$. And then from (8), we have

$$n^2 = 1-k^2, \qquad p_1 = 2-k^2, \qquad q_1 = -1,$$
$$\beta n^2 = -1, \qquad p_2 = -(1+m^2), \qquad q_2 = m^2. \tag{43}$$

Substituting (43) into (10), the parameters can be determined as

$$\frac{a^2}{c^2}=\sqrt{\frac{m^2}{1-k^2}},\qquad \frac{b}{a}=-[a^2(2-k^2)-3c^2(1+m^2)],$$
$$\frac{d}{c}=-[3a^2(2-k^2)-c^2(1+m^2)],\qquad \alpha=\pm\left[\frac{1-k^2}{m^2}\right]^{\frac{1}{4}}, \tag{44}$$

then the solution to the positive mKdV equation (4) is

$$\phi_{11}=\pm\left[\frac{1-k^2}{m^2}\right]^{\frac{1}{4}}[\mathrm{nd}(\xi,k)\mathrm{ns}(\eta,m)]. \tag{45}$$

Case 12. When $U=\mathrm{nd}(\xi,k)$ and $V=\mathrm{dc}(\eta,m)=\frac{\mathrm{dn}(\eta,m)}{\mathrm{cn}(\eta,m)}$. And then from (8), we have

$$n^2=1-k^2,\qquad p_1=2-k^2,\qquad q_1=-1,$$
$$\beta n^2=-1,\qquad p_2=-(1+m^2),\qquad q_2=m^2. \tag{46}$$

Substituting (46) into (10), the parameters can be determined as

$$\frac{a^2}{c^2}=\sqrt{\frac{m^2}{1-k^2}},\qquad \frac{b}{a}=-[a^2(2-k^2)-3c^2(1+m^2)],$$
$$\frac{d}{c}=-[3a^2(2-k^2)-c^2(1+m^2)],\qquad \alpha=\pm\left[\frac{1-k^2}{m^2}\right]^{\frac{1}{4}}, \tag{47}$$

then the solution to the positive mKdV equation (4) is

$$\phi_{12}=\pm\left[\frac{1-k^2}{m^2}\right]^{\frac{1}{4}}[\mathrm{nd}(\xi,k)\,\mathrm{dc}(\eta,m)]. \tag{48}$$

Case 13. When $U=\mathrm{cd}(\xi,k)$ and $V=\mathrm{dn}(\eta,m)$. And then from (8), we have

$$n^2=-k^2,\qquad p_1=-(1+k^2),\qquad q_1=1,$$
$$\beta n^2=1,\qquad p_2=2-m^2,\qquad q_2=-(1-m^2). \tag{49}$$

Substituting (49) into (10), the parameters can be determined as

$$\frac{a^2}{c^2}=\sqrt{\frac{1-m^2}{k^2}},\qquad \frac{b}{a}=[a^2(1+k^2)-3c^2(2-m^2)],$$
$$\frac{d}{c}=[3a^2(1+k^2)-c^2(2-m^2)],\qquad \alpha=\pm\left[\frac{k^2}{1-m^2}\right]^{\frac{1}{4}}, \tag{50}$$

then the solution to the positive mKdV equation (4) is

$$\phi_{13}=\pm\left[\frac{k^2}{1-m^2}\right]^{\frac{1}{4}}[\mathrm{cd}(\xi,k)\,\mathrm{dn}(\eta,m)]. \tag{51}$$

Comparing figure 5 with figure 1, we can see that even for solutions with different analytical expressions, they can show quite similar characteristics.

Case 14. When $U=\mathrm{ns}(\xi,k)$ and $V=\mathrm{dn}(\eta,m)$. And then from (8), we have

$$n^2=-1,\qquad p_1=-(1+k^2),\qquad q_1=k^2,$$
$$\beta n^2=1,\qquad p_2=2-m^2,\qquad q_2=-(1-m^2). \tag{52}$$

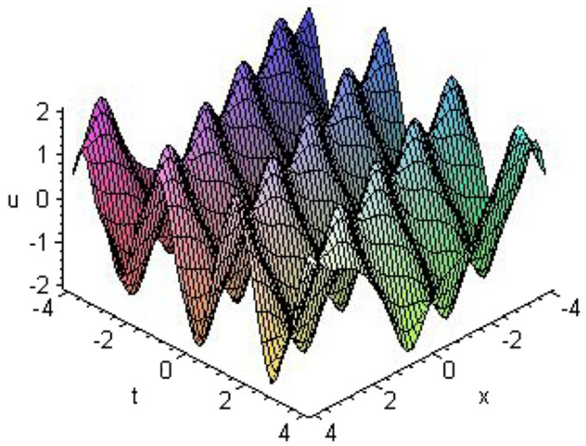


Figure 5. The graphical presentation shows the spacetime evolution of the breather lattice solution of equations (50) and (51), where the parameters are chosen as $a = 1$, $c = 1$, $m = 0.8$, $\xi_0 = \eta_0 = 0$, from which other parameters can be determined as $b = -2.72$, $d = 2.72$, $k = 0.6$ and $\alpha = 1$.

Substituting (52) into (10), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}}, \qquad \frac{b}{a} = [a^2(1+k^2) - 3c^2(2-m^2)],$$
$$\frac{d}{c} = [3a^2(1+k^2) - c^2(2-m^2)], \qquad \alpha = \pm\left[\frac{1}{k^2(1-m^2)}\right]^{\frac{1}{4}}, \tag{53}$$

then the solution to the positive mKdV equation (4) is

$$\phi_{14} = \pm\left[\frac{1}{k^2(1-m^2)}\right]^{\frac{1}{4}} [\mathrm{ns}(\xi,k)\,\mathrm{dn}(\eta,m)]. \tag{54}$$

Case 15. When $U = \mathrm{dc}(\xi, k)$ and $V = \mathrm{dn}(\eta, m)$. And then from (8), we have

$$n^2 = -1, \qquad p_1 = -(1+k^2), \qquad q_1 = k^2,$$
$$\beta n^2 = 1, \qquad p_2 = 2-m^2, \qquad q_2 = -(1-m^2). \tag{55}$$

Substituting (55) into (10), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}}, \qquad \frac{b}{a} = [a^2(1+k^2) - 3c^2(2-m^2)],$$
$$\frac{d}{c} = [3a^2(1+k^2) - c^2(2-m^2)], \qquad \alpha = \pm\left[\frac{1}{k^2(1-m^2)}\right]^{\frac{1}{4}}, \tag{56}$$

then the solution to the positive mKdV equation (4) is

$$\phi_{15} = \pm\left[\frac{1}{k^2(1-m^2)}\right]^{\frac{1}{4}} [\mathrm{dc}(\xi,k)\,\mathrm{dn}(\eta,m)]. \tag{57}$$

Case 16. When $U = \mathrm{nc}(\xi, k)$ and $V = \mathrm{nc}(\eta, m)$. And then from (8), we have

$$n^2 = -(1-k^2), \qquad p_1 = 2k^2 - 1, \qquad q_1 = -k^2,$$
$$\beta n^2 = -(1-m^2), \qquad p_2 = 2m^2 - 1, \qquad q_2 = -m^2. \tag{58}$$

Substituting (58) into (10), the parameters can be determined as

$$\frac{a^2}{c^2}=\frac{m}{k}\sqrt{\frac{1-m^2}{1-k^2}},\qquad \frac{b}{a}=-[a^2(2k^2-1)+3c^2(2m^2-1)],$$
$$\frac{d}{c}=-[3a^2(2k^2-1)+c^2(2m^2-1)],\qquad \alpha=\pm\left[\frac{(1-k^2)(1-m^2)}{k^2m^2}\right]^{\frac{1}{4}}, \tag{59}$$

then the solution to the positive mKdV equation (4) is

$$\phi_{16}=\pm\left[\frac{(1-k^2)(1-m^2)}{k^2m^2}\right]^{\frac{1}{4}}[\mathrm{nc}(\xi,k)\mathrm{nc}(\eta,m)]. \tag{60}$$

Case 17. When $U=\mathrm{sd}(\xi,k)$ and $V=\mathrm{sd}(\eta,m)$. And then from (8), we have

$$n^2=k^2(1-k^2),\qquad p_1=2k^2-1,\qquad q_1=1,$$
$$\beta n^2=m^2(1-m^2),\qquad p_2=2m^2-1,\qquad q_2=1. \tag{61}$$

Substituting (61) into (10), the parameters can be determined as

$$\frac{a^2}{c^2}=\frac{m}{k}\sqrt{\frac{1-m^2}{1-k^2}},\qquad \frac{b}{a}=-[a^2(2k^2-1)+3c^2(2m^2-1)],$$
$$\frac{d}{c}=-[3a^2(2k^2-1)+c^2(2m^2-1)],\qquad \alpha=\pm[k^2m^2(1-k^2)(1-m^2)]^{\frac{1}{4}}, \tag{62}$$

then the solution to the positive mKdV equation (4) is

$$\phi_{17}=\pm[k^2m^2(1-k^2)(1-m^2)]^{\frac{1}{4}}[\mathrm{sd}(\xi,k)\mathrm{sd}(\eta,m)]. \tag{63}$$

Case 18. When $U=\mathrm{ds}(\xi,k)$ and $V=\mathrm{ds}(\eta,m)$. And then from (8), we have

$$n^2=-1,\qquad p_1=2k^2-1,\qquad q_1=-k^2(1-k^2),$$
$$\beta n^2=-1,\qquad p_2=2m^2-1,\qquad q_2=-m^2(1-m^2). \tag{64}$$

Substituting (64) into (10), the parameters can be determined as

$$\frac{a^2}{c^2}=\frac{m}{k}\sqrt{\frac{1-m^2}{1-k^2}},\qquad \frac{b}{a}=-[a^2(2k^2-1)+3c^2(2m^2-1)],$$
$$\frac{d}{c}=-[3a^2(2k^2-1)+c^2(2m^2-1)],\qquad \alpha=\pm\left[\frac{1}{k^2m^2(1-k^2)(1-m^2)}\right]^{\frac{1}{4}}, \tag{65}$$

then the solution to the positive mKdV equation (4) is

$$\phi_{18}=\pm\left[\frac{1}{k^2m^2(1-k^2)(1-m^2)}\right]^{\frac{1}{4}}[\mathrm{ds}(\xi,k)\,\mathrm{ds}(\eta,m)]. \tag{66}$$

3. Conclusion and discussions

In this paper, dependent and independent variable transformations are introduced to solve the positive mKdV equation by using the knowledge of elliptic equation and Jacobian elliptic functions. It is shown that besides the solutions expressed in terms of the different combinations of Jacobi elliptic functions, there are the solutions expressed in terms of elementary functions, which can be obtained in the above solutions in the limit cases where k and/or m take the values 0 and/or 1. However, not all the combinations of Jacobi elliptic functions are the solutions to the positive mKdV equation (4), only those that satisfy constraints (11) can be the solutions to the positive mKdV equation (1). Furthermore, when different independent variable transformations are adopted, there will be different results. For example, when we choose the independent variable transformation

$$\xi = ax + \frac{1}{a}t + \xi_0, \qquad \eta = ax - \frac{1}{a}t + \eta_0, \tag{67}$$

which is given in [2], some breather lattice solutions expressed in terms of Jacobi elliptic functions will be omitted. Under variable transformations mentioned above, all solutions can be expressed in terms of 12 basic Jacobi elliptic functions listed in this paper, there are only 18 combinations of Jacobi elliptic functions that can satisfy constraints (11).

Because the emphasis of this paper is laid on giving a systematical way to obtain many kinds of breather lattice solutions (including two-soliton lattice solutions), we do not touch on the stability of these solutions. Although we do not give the stability analysis to our solutions, from the results given by Kevrekidis PG *et al* [7, 8], we can say that not all the solutions given in our paper are unstable. Even though the solutions are unstable, they can be stabilized by ac driving and damping, this has been reported in Kevrekidis PG *et al* [7, 8].

We know that the mKdV equation is derived from many physical situations, therefore its solutions will benefit to explain numerous physical phenomena, such as jamming in traffic flow, fluid dynamics and plasmas and so on, which have been pointed out in Kevrekidis's works [7, 8]. Due to the wide applications of the positive mKdV equation, the analytical solutions given in this paper will be helpful in related research.

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